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A Relativistic Gauge Theory of Nonlinear Quantum Mechanics and Newtonian Gravity

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Abstract A new kind of gauge theory is introduced, where the minimal coupling and corresponding covariant derivatives are defined in the space of functions pertaining to the functional Schrödinger picture of a given field theory. While, for simplicity, we study the example of a $\mathcal{U}(1)$ symmetry, this kind of gauge theory can accommodate other symmetries as well. We consider the resulting relativistic nonlinear extension of quantum mechanics and show that it incorporates gravity in the $(0+1)$-dimensional limit, similar to recently studied Schrödinger-Newton equations. Gravity is encoded here into a universal nonlinear extension of quantum theory. A probabilistic interpretation (Born's rule) holds, provided the underlying model is scale free.

Keywords Nonlinear functional Schrödinger equation · Gauge symmetry · Newtonian gravity

1 Introduction

Linearity of the (functional) Schrödinger equation has been an essential and a puzzling ingredient of quantum (field) theory since its earliest days. Practically all physical phenomena show nonlinear behaviour when examined over a sufficiently large range of the dynamical parameters that determine an evolving object. What singles out the linear dynamics and validity of the superposition principle for the wave function(al)? Quantum mechanics is very successfully tested experimentally under a wide range of laboratory conditions. Yet the mathematical structure of the theory, so far, hinges heavily on the linear structure embodied in linear operators acting on states represented by rays in a Hilbert space [1, 2].

This raises the question: Are nonlinear extensions possible which agree with the standard formulation in its experimentally ascertained domain of validity?

If so, could this alleviate the unresolved measurement problem [1–3]? While the outcome of this second question is still open, it seems worth while to mention that in recent studies of

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the necessarily related wave function collapse or reduction mechanisms by Pearle [4] and by Bassi [5] the authors indicate that a nonlinear extension of quantum theory might ultimately account consistently for these effects.

Our present aim is to report on a universal nonlinear extension of quantum field theory based on a new *functional gauge symmetry*, which operates on the space of field configurations rather than on the underlying spacetime [6, 7]. In particular, we will argue that this theory essentially incorporates Newtonian gravity, which invites deliberation whether such an approach could be of wider use. Gravity, in this picture, appears as a manifestation of the nonlinearity of quantum mechanics.

Among the numerous earlier works that have attempted to extend quantum theory in a nonlinear way, we should like to mention these: The work by Kibble and by Kibble and Randjbar-Daemi is close to ours in that they consider how nonlinear modifications of quantum field theory can be made compatible with Lorentz or more generally coordinate invariance [8, 9]. Besides considering a coupling of quantum fields to classical gravity according to general relativity, which induces an intrinsic nonlinearity [9, 10], these authors study mean-field type nonlinearities, where parameters of the model are state dependent through their assumed dependence on expectations of certain operators. The work by Bialynicki-Birula and Mycielski introduces a logarithmic nonlinearity into the nonrelativistic Schrödinger equation, which has the advantage that many of the nice features of quantum mechanics are left intact [11]. A number of different nonrelativistic models of this kind have been systematically studied by Weinberg, offering also an assessment of the observational limits on such modifications of the Schrödinger equation [12, 13].

Independently, Doebner and Goldin and collaborators have also studied nonlinear modifications of the nonrelativistic Schrödinger equation [14–17]. While this was originally motivated by attempts to incorporate dissipative effects, they later have shown that classes of nonlinear Schrödinger equations, including many of those considered in the literature, for example, the one proposed in [11], can be obtained through nonlinear (in the wave function) transformations of the linear quantum mechanical equation. They coined the name "gauge transformations of the third kind" in this context, in analogy with the reasoning for gauge transformations of the second kind (corresponding to the usual minimal coupling). In distinction to their work, our functional gauge transformations, being set up for quantum field theory, work on the field configuration space over which the *wave functional* is defined. This will be most clearly recognized in the way we introduce *covariant* functional derivatives (cf. (12), (13) in Sect. 3). (Of course, the fact that functional derivatives come into play here is not new *per se*: they are to the functional Schrödinger picture of quantum field theory developed earlier by Jackiw and collaborators—reviewed and generalized for fermions in [18]—what ordinary derivatives are to quantum mechanics.)

The necessity of generalizing quantum dynamics for quantum gravity has been discussed in view of the "problem of time" and the Wheeler-DeWitt equation by Kiefer and by Barbour [19, 20]. We recall that this equation, playing the role of the Schrödinger equation there, is of the form of a constraint operator, i.e. the Hamiltonian of canonical gravity, acting on the wave functional, $\hat{H}\Psi = 0$. There are two unpleasant features: no time derivative appears [10, 19] and, since $\hat{H}$ is Hermitean, there seems to be no indication of complex solutions [20]! Therefore, both authors pointed out that nonlinear modifications would be a welcome remedy and in [19] it was proposed that these may assume the form of a "supergauge potential" defined on configuration space. While formally analogous to the gauge connection in the covariant derivatives introduced here, only some preliminary interpretation has been offered that such connection might effectively represent certain quantum (vacuum) effects of matter.

In distinction, based on the proposed functional gauge symmetry, all dynamical and constraint equations of our theory are consistently derived from a gauge and Lorentz invariant action, as we shall discuss. From the outset, this has nothing to do with gravity, in particular, but may be applied to any quantum field theory.

The importance of maintaining a probabilistic interpretation of the wave function following the Born rule is stressed in all previous works. We will recover this as well. However, no understanding of the origins of the proposed nonlinearities has been provided, except in the obvious case of gravity coupling studied by Kibble and Randjbar-Daemi [9]. Presently, this is achieved by the gauge principle. Furthermore, we obtain the surprising result that our theory automatically incorporates gravity in its simplest Newtonian form, which will be discussed in Sect. 6.

Part of the motivation for the present work comes from recent considerations of the possibility of a deterministic foundation of quantum mechanics, as it has already been verified in a number of models [21–39]. While general principles and physical mechanisms ruling the construction of a deterministic classical model underlying a given quantum field theory are hard to come by, cf. [26], the already known models are quite promising, amounting to an existence proof that the quantum harmonic oscillator can be understood completely in classical deterministic terms, see [21–25, 33, 34].

In general, we expect that with improved understanding of the emergence of quantum mechanics also resulting nonlinear corrections to quantum mechanics, as it is, should become visible. Note that any model which has an evolution equation that is linear in the wave function, i.e. without any nonlinear feedback, can always be cast into the form of a Schrödinger equation, possibly with modified potentials etc. Nonlinearity seems unavoidable, if one wants to go beyond the canonical framework of quantum theory. In the present work, such a nonlinear extension of quantum field theory is a central aspect.

The paper is organized as follows. In Sect. 2, we recapitulate the work of [9] which forms the basis for our argument that the gauge invariant (quantum) action introduced in Sect. 3 can be written in a Lorentz invariant way, despite the presence of a fundamental length parameter. In Sect. 4, the dynamical and constraint equations are presented and a crucial "nonlinearity factor" of the action is determined. Section 5 is dedicated to the discussion of the validity of the Born rule in the resulting nonlinear quantum theory. In Sect. 6, it is demonstrated that it leads to the Schrödinger-Newton equations in the one-dimensional limit considering stationary states. Concluding remarks follow in Sect. 7.

2 The Schrödinger Picture for Given Background Space-Time

Following the work of Kibble and Randjbar-Daemi [9], we consider a four-dimensional globally hyperbolic manifold $\mathcal{M}$ with a *given* metric $g_{\mu\nu}$ of signature $(1,-1,-1,-1)$.[1] Then, it is always possible to introduce a global slicing into space-like hypersurfaces, such that a chosen family of such surfaces, $\{\sigma(t)\}$, is locally determined by:

$$x^\mu = x^\mu(\xi^1, \xi^2, \xi^3; t), \tag{1}$$

in terms of intrinsic coordinates ξ^r, and there exists an everywhere time-like vectorfield n^μ, the normal, with $n_\mu n^\mu = 1$ and $n_\mu x^\mu_{,r} = 0$, where $x^\mu_{,r} \equiv \partial x^\mu/\partial \xi^r$. We will make use of the derivative with respect to t at fixed ξ^r of a function f, $\dot{f} \equiv \partial f/\partial t|_\xi$. In particular, then, the

[1] Units are chosen such that $\hbar = c = 1$.

lapse function N and shift vector N^r are introduced through the relation $\dot{x}^\mu = Nn^\mu + N^r x^\mu_{,r}$, the geometrical meaning of which is illustrated, for example, in Chap. 3.3 of [10].

We assume a given Lagrangean L of a field theory, such as for a real scalar field ϕ:

$$L \equiv \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{2}$$

where $V(\phi)$ incorporates mass or selfinteraction terms. Then, the invariant action is defined by:

$$S \equiv \int \mathrm{d}^4 x \sqrt{-g} L, \tag{3}$$

where $g \equiv \det g_{\mu\nu}$. This, in turn, yields the stress-energy tensor $T^{\mu\nu}$ through the relation:

$$\frac{1}{2}\sqrt{-g} T^{\mu\nu} \equiv \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2}\sqrt{-g}(\partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} L). \tag{4}$$

With the help of the induced metric γ_{rs} on $\sigma(t)$, $\gamma_{rs} \equiv g_{\mu\nu} x^\mu_{,r} x^\nu_{,s}$, and the hypersurface element $\mathrm{d}\sigma_\mu \equiv \mathrm{d}^3\xi \sqrt{-\gamma} n_\mu$, the surface-dependent Hamiltonian can be defined:

$$H(t) \equiv \int_{\sigma(t)} \mathrm{d}\sigma_\mu T^\mu_\nu \dot{x}^\nu. \tag{5}$$

In the simplest case, with $\dot{x}^\mu = \delta^\mu_0$ (i.e., $N = 1$, $N^r = 0$) and $x^\mu_{,r} = \delta^\mu_r$, these relations reduce to $\gamma_{rs} = g_{rs}$ and $H(t) = \int_{\sigma(t)} \mathrm{d}^3\xi T^{00}$, as expected.

If the stress-energy tensor can be expressed in terms of canonical coordinates and momenta, for example, the scalar field $\phi = \phi(\xi^i, t)$ and its conjugated momentum $\Pi = \Pi(\xi^i, t)$ on time slices $\sigma(t)$, we assume that the corresponding quantized theory exists, with ϕ and Π fulfilling the usual equal-t commutation relation. Of course, matters are not that simple in a general curved background. Therefore, a heuristic derivation of the Schrödinger picture from the manifestly covariant Heisenberg picture has been presented in [9]. We will not pursue this further, since our aim here is simply to recover their Lorentz invariant form of the functional Schrödinger equation, a generalization of which will follow from the action principle to be considered in the course of this work.[2]

In any case, the functional Schödinger equation obtained by Kibble and Randjbar-Daemi appears naturally as one would guess ($\hbar = c = 1$):

$$i\dot{\Psi} = H(t)\Psi. \tag{6}$$

Using the surface element $\mathrm{d}\sigma_\mu$ given above, together with (5), and:

$$\dot{\Psi} = \int_{\sigma(t)} \mathrm{d}^3\xi \dot{x}^\mu \frac{\delta}{\delta x^\mu} \Psi, \tag{7}$$

the Schrödinger equation can also be represented in a local form:

$$i \frac{\delta}{\delta x^\mu} \Psi = \sqrt{-\gamma} n_\nu T^\nu_\mu \Psi. \tag{8}$$

[2] As remarked in [9], the derivation from an action principle will guarantee the general coordinate invariance of the theory. However, the Schrödinger picture clearly depends on the slicing of space-time as well as on the parametrisation of the slices. Thus, invariance under surface deformations—which can be restricted to diffeomorphism invariance [10]—is not implied here.

We take from this section that the functional Schrödinger equation can be written in a way that makes the behaviour under Lorentz transformations explicit. This applies, in particular, to the case of a flat background space-time, where field quantization is well understood.

3 The Gauge Invariant Action

We consider the generic scalar field theory described by the Lagrangean of (2), while internal symmetries and fermions can be introduced as we discussed earlier in the second of [6, 7]. Furthermore, specializing the result of the previous section for *Minkowski space*, we find:

$$H(t) = \int_{\sigma(t)} \mathrm{d}^3\xi\, T^{00} = \int \mathrm{d}^3x \left\{ -\frac{1}{2}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right\} \equiv H[\hat{\pi}, \phi], \tag{9}$$

i.e., the usual Hamiltonian which is independent of the parameter time t; intrinsic and Minkowski space coordinates have been identified, $\vec{\xi} = \vec{x}$.

Here, in the Hamiltonian of (9) in particular, quantization is implemented by substituting the canonical momentum π conjugate to the field ϕ (playing the role of "coordinate"):

$$\pi(\vec{x}) \longrightarrow \hat{\pi}(\vec{x}) \equiv \frac{1}{i}\frac{\delta}{\delta\phi(\vec{x})}. \tag{10}$$

Correspondingly, we have $\Psi = \Psi[\phi; t]$, i.e. a time dependent functional, in this coordinate representation, and $\dot{\Psi} = \partial_t \Psi$. So far, this is the usual functional Schrödinger picture of quantum field theory applied to the chosen example of a scalar model [18, 40–45].

Next, we introduce *functional gauge transformations* [6, 7]:

$$\Psi'[\phi; t] = \exp(i\Lambda[\phi; t])\Psi[\phi; t], \tag{11}$$

where Λ denotes a time dependent real functional. These $\mathcal{U}(1)$ transformations are local in the space of field configurations. They differ from the usual gauge transformations in QFT, since we introduce covariant derivatives by the following replacements:

$$\partial_t \longrightarrow \mathcal{D}_t \equiv \partial_t - i\mathcal{A}_t[\phi; t], \tag{12}$$

$$\frac{\delta}{\delta\phi(\vec{x})} \longrightarrow \mathcal{D}_{\phi(\vec{x})} \equiv \frac{\delta}{\delta\phi(\vec{x})} - i\mathcal{A}_\phi[\phi; t, \vec{x}]. \tag{13}$$

The real functional $\mathcal{A}$ presents a new kind of 'potential' or 'connection'. Generally, $\mathcal{A}$ depends on t. However, it is a *functional* of ϕ in (12), while it is a *functional field* in (13). We distinguish these components of $\mathcal{A}$ by the subscripts. Furthermore, the 'potentials' are required to transform as:

$$\mathcal{A}'_t[\phi; t] = \mathcal{A}_t[\phi; t] + \partial_t\Lambda[\phi; t], \tag{14}$$

$$\mathcal{A}'_\phi[\phi; t, \vec{x}] = \mathcal{A}_\phi[\phi; t, \vec{x}] + \frac{\delta}{\delta\phi(\vec{x})}\Lambda[\phi; t]. \tag{15}$$

Applying (11–15), it follows that the correspondingly generalized functional Schrödinger equation is invariant under the $\mathcal{U}(1)$ gauge transformations.

Furthermore, it is suggestive to introduce an invariant 'field strength':

$$\mathcal{F}_{t\phi}[\phi; t, \vec{x}] \equiv \partial_t\mathcal{A}_\phi[\phi; t, \vec{x}] - \frac{\delta}{\delta\phi(\vec{x})}\mathcal{A}_t[\phi; t], \tag{16}$$

in close analogy to ordinary gauge theories; note that $\mathcal{F}_{t\phi} = [\mathcal{D}_t, \mathcal{D}_\phi]/(-i)$.

We now postulate a consistent dynamics for the gauge 'potential' $\mathcal{A}$, in order to give a meaning to the above 'minimal coupling' prescription. All elementary fields supposedly are present as the coordinates on which the wave functional depends—presently just a scalar field, besides time. We consider the following $\mathcal{U}(1)$ invariant action:

$$\Gamma \equiv \int \mathrm{d}t \mathrm{D}\phi \left\{ \Psi^* \left(\mathcal{N}(\rho)\, i \overleftrightarrow{\mathcal{D}}_t - H\left[\frac{1}{i}\mathcal{D}_\phi, \phi\right]\right)\Psi + \frac{l^2}{2}\int \mathrm{d}^3x (\mathcal{F}_{t\phi})^2 \right\}, \tag{17}$$

where $\Psi^* \mathcal{N}\, i \overleftrightarrow{\mathcal{D}}_t\, \Psi \equiv \frac{1}{2}\mathcal{N}\{\Psi^* i\mathcal{D}_t \Psi + (i\mathcal{D}_t\Psi)^*\Psi\}$, and with a dimensionless real function $\mathcal{N}$ which depends on the density:

$$\rho[\phi; t] \equiv \Psi^*[\phi; t]\Psi[\phi; t]. \tag{18}$$

We shall see shortly that $\mathcal{N}$ incorporates a *necessary nonlinearity*; it will be uniquely determined in Sect. 4, cf. (25). The fundamental parameter l has dimension $[l] = [length]$, for dimensionless measure $\mathrm{D}\phi$ and Ψ, independently of the dimension of space-time.[3]

The action Γ generalizes the action for the wave functional of a scalar field, which has been employed for applications of Dirac's variational principle to QFT, e.g., in [9, 40–43]. The quadratic part in $\mathcal{F}_{t\phi}$ is the simplest possible extension, i.e. local in ϕ and quadratic in the derivatives, together with the nonlinearity $\mathcal{N}(\rho)$ introduced here.

An immediate consequence of the $\mathcal{U}(1)$ invariance is that the Hamiltonian H, unlike in QFT, cannot be arbitrarily shifted by a constant ΔE, gauge transforming $\Psi \to \exp(-i\Delta E t)\Psi$. Thus, there is an absolute meaning to the zero of energy in this theory.

Translation invariance of the action, (17), gives rise to a *conserved energy functional*, where a contribution which is solely due to $\mathcal{A}_t$ and $\mathcal{A}_\phi$ is added to the matter term, which is modified by the covariant derivatives.

Furthermore, following from the discussion of Sect. 2, the Lorentz invariance of this theory is guaranteed. In particular, the action can be written in a Lorentz (and Poincaré) invariant way, using the appropriate surface-dependent Hamiltonian, cf. (5), despite that a fundamental length l enters here.[4]

The action depends on Ψ, Ψ^*, $\mathcal{A}_t$, and $\mathcal{A}_\phi$ separately. While a Hamiltonian formulation is possible, the equations of motion and a constraint can be obtained directly by varying Γ with respect to these variables.

4 The Dynamical and Constraint Equations

The dynamical equations of motion were previously obtained in [6, 7] and are reproduced here for convenience. The gauge covariant equation for the Ψ-functional is:

$$(\rho\mathcal{N}(\rho))' i\mathcal{D}_t \Psi[\phi; t] = H\left[\frac{1}{i}\mathcal{D}_\phi, \phi\right]\Psi[\phi; t], \tag{19}$$

[3]Note that this parameter has *necessarily* the dimension of a length, in order to give the action its correct dimension; it presents the coupling constant of our theory and will be related to Newton's constant in Sect. 6. By suitably rescaling the gauge 'potentials', the coupling constant could be moved to the covariant derivatives, as originally discussed [6, 7]; however, as is familiar from ordinary non-Abelian gauge theories, the equivalent action is often more convenient where the coupling appears only in one place.

[4]The coordinates x^μ, of course, must not be confused with the intrinsic coordinates ξ^i and time parameter t.

where $f'(\rho) \equiv \mathrm{d}f(\rho)/\mathrm{d}\rho$. This replaces the usual functional Schrödinger equation (and similarly its adjoint).

The nonlinear (19) preserves the normalization of Ψ. Fixing it at an initial parameter time, in terms of an arbitrary constant $\mathcal{C}_0$:

$$\langle\Psi|\Psi\rangle \equiv \int \mathrm{D}\phi\Psi^*\Psi = \mathcal{C}_0, \tag{20}$$

it is conserved under further evolution, while the overlap of two different states, $\langle\Psi_1|\Psi_2\rangle$, may vary. This is indicative of the standard probability interpretation related to $\Psi^*\Psi$, which we will discuss in the next section in more detail.

Next, there is an invariant 'gauge field equation':

$$\partial_t\mathcal{F}_{t\phi}[\phi;t,\vec{x}] = -\frac{1}{2il^2}(\Psi^*[\phi;t]\mathcal{D}_{\phi(\vec{x})}\Psi[\phi;t] - \Psi[\phi;t](\mathcal{D}_{\phi(\vec{x})}\Psi[\phi;t])^*), \tag{21}$$

which completes the dynamical equations.

However, there is no time derivative acting on the variable $\mathcal{A}_t$ in the action. Therefore, it acts as a Lagrange multiplier for a constraint, which is the gauge invariant 'Gauss' law':

$$\int \mathrm{d}^3x \frac{\delta}{\delta\phi(\vec{x})}\mathcal{F}_{t\phi}[\phi;t,\vec{x}] = -\frac{1}{l^2}\rho\mathcal{N}(\rho). \tag{22}$$

Of course, it differs from the usual one in QED, for example. This raises the question, whether our functional $\mathcal{U}(1)$ gauge symmetry is compatible with the presence of standard internal symmetries. This is answered affirmatively in the second of [6, 7].

Equation (22) can be combined with (21) to result in a continuity equation:

$$0 = \partial_t(\rho\mathcal{N}(\rho)) - \frac{1}{2i}\int \mathrm{d}^3x \frac{\delta}{\delta\phi(\vec{x})}(\Psi^*\mathcal{D}_{\phi(\vec{x})}\Psi - \Psi(\mathcal{D}_{\phi(\vec{x})}\Psi)^*), \tag{23}$$

expressing local $\mathcal{U}(1)$ 'charge' conservation in the space of field configurations. Functionally integrating (22), we find that the total 'charge' Q has to vanish at all times:

$$Q(t) \equiv \frac{1}{l^2}\int \mathrm{D}\phi\rho\mathcal{N}(\rho) = 0, \tag{24}$$

since the functional integral of a total derivative is zero [44, 45]. This is different from integrating the usual Gauss' law in electrodynamics over all space, for example, were there can be a flux of the fields out to infinity. The necessity of the nonlinearity now becomes obvious. Without it, the vanishing total 'charge' could not be implemented, as it would be in conflict with the normalization (20).

We proceed to determine the nonlinearity factor, $\mathcal{N}(\rho) \neq 1$. In fact, we would like to implement (24), similarly as the normalization, at an initial parameter time t. Since it has to be a constant of motion, $\partial_t Q(t) = 0$, we express this, with the help of (19), as a condition on $\rho\mathcal{N}(\rho)$. It is easily seen that the only solution here is a linear function:

$$\rho\mathcal{N}(\rho) = \mathcal{C}_1\left(\rho - \mathcal{C}_0\left(\int \mathrm{D}\phi\right)^{-1}\right), \tag{25}$$

if one wants to avoid further constraining Ψ or Ψ^*; the latter would make it more difficult, if not impossible, to obtain linear quantum mechanics as a limiting case.[5]

Evidently, the volume of the space of fields, $\Omega \equiv \int \mathrm{D}\phi$, needs to be regularized, as well as the second functional derivatives at coinciding points which appear. A cut-off on field amplitudes has to be introduced together, for example, with dimensional regularization [44, 45] or, more convenient here, the point-splitting technique [18]. Clearly, a related renormalization procedure is an interesting subject for further study, since it has to take into account the new functional gauge symmetry.

5 Probability Versus 'Charge'

The *homogeneity property* is necessary for the probability interpretation of the density $\rho = \Psi^*\Psi$ according to Born's rule [8, 11–13]: Ψ and $z\Psi$ ($z \in \mathbf{Z}$) have to present the same physical state. In this way, states are associated with rays in a Hilbert space (instead of vectors).

For the present case, it is useful to consider the set of scale transformations:

$$\rho = \mathcal{C}_0^a \mathcal{C}_1^{-1} \rho', \qquad \int \mathrm{D}\phi = \mathcal{C}_0^{1-a} \mathcal{C}_1 \int \mathrm{D}\phi', \tag{26}$$

such that $\int \mathrm{D}\phi' \rho' = 1$; we recall that the real measure $\mathrm{D}\phi$ and constants $\mathcal{C}_{0,1}$ are chosen dimensionless, without loss of generality; a is real. Furthermore, we rescale:

$$(\vec{x}; t) = \mathcal{C}_0^{-a/2} \mathcal{C}_1^{1/2} (\vec{x}'; t'), \qquad (\phi; \mathcal{A}_t) = \mathcal{C}_0^{a/2} \mathcal{C}_1^{-1/2} (\phi'; \mathcal{A}_t'), \tag{27}$$

and, consistently:

$$(\delta_\phi; \mathcal{A}_\phi) = \mathcal{C}_0^a \mathcal{C}_1^{-1} (\delta_{\phi'}; \mathcal{A}_{\phi'}). \tag{28}$$

Under these transformations, the action transforms as:

$$\Gamma = \mathcal{C}_1 \Gamma', \tag{29}$$

where Γ' is defined like Γ, (17), however, replacing all quantities by the primed ones. One arrives at this result, provided the Hamiltonian H, cf. (9), contains *no dimensionful constants*, such as in a Lagrangean mass term, $\propto m^2\phi^2$, which could be contained in our model; a selfinteraction of the form $\propto \lambda\phi^4$ introduces a dimensionless coupling λ instead.

There are several implications. First, the scale transformations change the overall scale of the action, say, in units of $\hbar$, by the constant factor $\mathcal{C}_1$. This is equivalent to the rescaling $\hbar = \hbar'/\mathcal{C}_1$. However, since we prefer to choose units such that $\hbar = 1$, we should also fix $\mathcal{C}_1 = 1$, henceforth.[6] Second, since the constant $\mathcal{C}_0$ does not affect the transformation of Γ, we can always choose to normalize the wave functional to $\mathcal{C}_0 = 1$, see (20).

We see that states, as far as Ψ is concerned, are represented by rays. Therefore, a probability interpretation of $\Psi^*\Psi$ according to the *Born rule* can be maintained. This is in agreement with the observation that (19), if it were not for the presence of the covariant derivatives, now appears like the usual functional Schrödinger equation. Summarizing the previous

[5] In [7], a logarithmic form was chosen. In view of the present discussion, however, it should be dismissed, since it is not based on a constant of motion.

[6] The overall sign of $\rho\mathcal{N}(\rho)$, cf. (25), is chosen with hindsight, see Sect. 6.

discussion, we now have:

$$\rho\mathcal{N}(\rho) = \rho - \left(\int \mathrm{D}\phi\right)^{-1}, \tag{30}$$

$$i\mathcal{D}_t\Psi[\phi;t] = H\left[\frac{1}{i}\mathcal{D}_\phi, \phi\right]\Psi[\phi;t]. \tag{31}$$

However, it must be stressed that the 'potentials' $\mathcal{A}_t$ and $\mathcal{A}_\phi$ are selfconsistently determined through (21), (22). Therefore, we arrive here at intrinsically *nonlinear quantum mechanics*.[7]

The difference to standard quantum mechanics also shows up clearly in (23), with the first term now replaced by $\partial_t\rho$: the flux of probability over the space of field configurations is affected nonlinearly by Ψ^* and Ψ through the 'potentials'.

Finally, we remark that *in presence of dimensionful parameters* in the Hamiltonian the above scale symmetry, (26–29), breaks down. In particular, then the normalization of Ψ cannot be chosen freely; correspondingly, rays break into inequivalent vectors. In this situation, it is appropriate to consider Ψ and Ψ^* as giving rise to two oppositely 'charged' real components of the wave functional, $\Psi_+ \equiv (\Psi + \Psi^*)/\sqrt{2}$ and $\Psi_- \equiv (\Psi - \Psi^*)/i\sqrt{2}$, which interact, while preserving the normalization of $\Psi^*\Psi$. Different normalizations, then, correspond to physically different sectors of the theory, i.e. a charge superselection rule.

However, the *absence of the homogeneity property* modifies the usual measurement theory. In particular, the usual "reduction of the wave packet" postulate [1] cannot be maintained, in this case. This has been discussed in detail in [8] and formed the starting point for the particular nonlinear theory proposed there, mentioned before in Sect. 1.

6 Stationary States and the Schrödinger-Newton Equations

The time dependence in (19–22) can be separated with the Ansatz $\Psi[\phi;t] \equiv \exp(-i\omega t)\Psi_\omega[\phi]$, $\omega \in \mathbf{R}$, and consistently assuming *time independent* $\mathcal{A}$-functionals. Thus, the (19), together with (30), yields:

$$\omega\Psi_\omega[\phi] = H\left[\frac{1}{i}\mathcal{D}_\phi, \phi\right]\Psi_\omega[\phi] - \mathcal{A}_t[\phi]\Psi_\omega[\phi], \tag{32}$$

with $\mathcal{D}_\phi = \frac{\delta}{\delta\phi} + i\mathcal{A}_\phi$ and $\rho_\omega \equiv \Psi^*_\omega[\phi]\Psi_\omega[\phi]$. From (21) follows:

$$\frac{1}{2i}(\Psi^*_\omega[\phi]\mathcal{D}_{\phi(\vec{x})}\Psi_\omega[\phi] - \Psi_\omega[\phi](\mathcal{D}_{\phi(\vec{x})}\Psi_\omega[\phi])^*) = 0, \tag{33}$$

which expresses the vanishing of the 'current' in the stationary situation. Applying a time independent gauge transformation, cf. (11), (15), the stationary wave functional can be made

[7] In the second of [7], we have argued that *microcausality* of the present theory holds. The *weak superposition principle* [11], generally, must be expected to fail: for two non-overlapping sources adding to the right-hand sides of (21), (22), the resulting 'potentials' must be expected to propagate away from the sources in field space. Thus, the sum of two non-overlapping solutions $\Psi_{1,2}$ will hardly present a solution of the coupled equations. However, *if* two *stationary non-overlapping solutions* exist, *then* their sum also presents a solution; see the stationary equations in Sect. 6.

real. Then, (33) implies $\mathcal{A}_\phi = 0$; consequently, $\mathcal{D}_\phi \to \frac{\delta}{\delta\phi}$ everywhere. Finally, 'Gauss' law', (22), determines $\mathcal{A}_t$:

$$\int \mathrm{d}^3x \frac{\delta^2}{\delta\phi(\vec{x})^2}\mathcal{A}_t[\phi] = \frac{1}{l^2}\left(\rho - \left(\int \mathrm{D}\phi\right)^{-1}\right), \tag{34}$$

which has to be solved selfconsistently together with (32). Separation of the time dependence thus leads to two coupled equations. They represent a field theoretic generalization of the stationary Schrödinger-Newton equations, as we shall explain.

The time dependent *Schrödinger-Newton equations* for a particle of mass m are given by:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - m\Phi\psi, \qquad \nabla^2\Phi = 4\pi G m|\psi|^2, \tag{35}$$

where $G \equiv l_P^2 c^2/\hbar$ is Newton's gravitational constant (here related to the Planck length l_P) and Φ denotes the gravitational potential. They represent the nonrelativistic approximation to "semiclassical gravity", i.e. Einstein's field equations coupled to the expectation value of the operator-valued stress-energy tensor of quantum matter—see, for example, [9, 46–50], and further references therein. They play an important role in arguments related to "semiclassical gravity", to gravitational self-localization of mesoscopic or macroscopic mass distributions, and to the role of gravity in Diósi's and Penrose's objective reduction scenarios.

Considering a Universe which consists only of a single point, we find that our field theory equations (32) and (34) reduce to the stationary Schrödinger-Newton equations *in one dimension*. Appropriate rescalings by powers of l, m, $\hbar$, and c of the various quantities have to be incorporated, in order to give the equations their one-dimensional form, where Newton's constant G is a dimensionless parameter. If there is a nonzero potential $V(\phi)$ in our Hamiltonian, this extends the Schrödinger equation in (35) by an additional potential term. Explicitly, keeping units such that $\hbar = c = 1$ and considering the Hamiltonian of (9) with $V(\phi) \equiv 0$, the following substitutions have to be performed, in order to arrive at the stationary limit of (35):

$$|\Psi|^2 \longrightarrow 4\pi G^2 m|\psi|^2, \qquad \mathcal{A}_t \longrightarrow m\Phi, \tag{36}$$

$$\begin{aligned} &\int \mathrm{d}^3x \frac{\delta^2}{\delta\phi(x)^2} \longrightarrow \frac{1}{m}\frac{\mathrm{d}^2}{\mathrm{d}q^2}, \\ &\int \mathcal{D}\phi \longrightarrow (4\pi G^2 m)^{-1}\int_{-Q/2}^{+Q/2} dq = Q/4\pi G^2 m, \end{aligned} \tag{37}$$

where m is the relevant (particle) mass scale and Q denotes a regulator length, much larger than any length scale of the one-dimensional system. Of course, the gradient terms of the Hamiltonian, $\propto (\nabla\phi)^2$, do not contribute in this limit ("a single point has no neighbours").

It seems remarkable that the gravitational interaction arises here in the space of quantum states (configuration space). Yet, in view of the fundamental length l present in the action, (17), it is perhaps not a complete surprise that our gauge theory incorporates gravity. We notice, however, also a *deviation from Newtonian gravity*, presented by the constant term on the right-hand side of (34). While it is natural to let this term become arbitrarily small in the quantum mechanical limit just discussed, its presence was shown to be necessary for the full theory in Sect. 4. This is an important topic for further study, related to the regularization of the theory.

In [49], it has recently been shown that sufficiently large Gaussian wave packets show a tendency to shrink in width as they evolve according to the time dependent Schrödinger-Newton equations. This leads to a *decrease of interference effects*, which possibly will be observable in near-future molecular interference experiments. It will be interesting to study the behaviour of such wave packets according to the present theory. We speculate that coherent superpositions of displaced wave packets (*Schrödinger cat states*) will decay by giving rise to time dependent 'potentials' $\mathcal{A}_t$ and $\mathcal{A}_\phi$, while attracting each other similar to corresponding classical matter distributions. If these remarks can be further substantiated, this should have some impact on further attempts to understand the "collapse of the wave function" or "reduction of the wave packet" in a consistent dynamical theory.

7 Concluding Remarks

A relativistic $\mathcal{U}(1)$ gauge theory has been presented which constitutes an intrinsically nonlinear extension of quantum mechanics or quantum field theory.

Closest in spirit seems the work of Kibble and Randjbar-Daemi [9] where such nonlinearities—due to coupling the expectation of the quantum matter stress-energy tensor to classical general relativity or to making parameters of the theory state dependent—have been discussed in a relativistic setting before. However, this has been reminiscent of a mean-field approximation.

In distinction, based on the gauge principle, we have introduced two 'potentials', $\mathcal{A}_t$ and $\mathcal{A}_\phi$, which are *not independent* new fields but functionals that depend on the same field variables of the underlying (scalar or other) field theory as the wave functional Ψ. The relevant dynamical and constraint equations follow from a relativistic invariant action principle, postulated in Sect. 3. Thus, if the 'potentials' are eliminated, in principle, by solving the respective equations, a nonlinear theory in Ψ necessarily results.

We observe that in the absence of quantum matter, $\Psi = 0$, the (21) and (22) that determine the 'field strength' $\mathcal{F}_{t\phi}$—and similarly in the $(0+1)$-dimensional limit—have no time dependent solutions. Therefore, the 'potentials' do not propagate independently of matter sources here.[8]

We have shown that the essential homogeneity property holds, which is related to the representation of states by rays in Hilbert space. Thus, the Born rule can be applied, giving a probabilistic interpretation to $\Psi^*\Psi$ [8, 11–13]. However, it breaks down, if the assumed underlying classical model contains dimensionful parameters. In this case, a discussion in terms of the 'charged' components of Ψ is appropriate, which invites further interpretation.

Related to the presence of a fundamental length l in the action, we have shown that in the zero-dimensional limit the presented theory recovers the recently much studied Schrödinger-Newton equations, coupling Newtonian gravity to quantum mechanics [9, 46–50]. Thus, the proposed theory incorporates Newtonian gravity into quantum field theory: *unlike independent degrees of freedom coupled to matter in the usual way, gravity is encoded here into a universal nonlinear extension of quantum field theory*.

In the future, the regularization of the theory and a perturbative scheme need to be worked out, in order to have control of its microscopic behaviour in situations where grav-

[8]This is due to the fact that the analogue of a magnetic field is missing for any underlying model based on a one-component field, see (2) and (16). The situation changes in the presence of internal symmetries, as discussed in the second of [7].

ity is weak. Naturally, it will be interesting whether the presented ideas of a new *functional gauge symmetry* can be further generalized and what ensuing experimental predictions will be.

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